Diffusion of ideas, social reinforcement and percolation

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***Abstract*—This paper analyzes how social structure and social reinforcement affect the diffusion of an idea in a population of human agents. A percolation approach is used to model the diffusion process. This framework assumes that information is local and embedded in a social network. We introduce social reinforcement in the model by softening the condition to adopt when the number of adopting neighbors increases. Our numerical analysis shows that social reinforcement severely affects the output of the process. Some ideas with an original value so low that it would never get diffused can be spread due to the strength of social reinforcement. This effect also interacts with the structure of the network, with a more sizeable impact on small worlds with a low rewiring probability. Also, social reinforcement completely changes the effect of clustering links, because sequential adoption of neighbors can make one agent adopt at later stages.**

I. Introduction

The success or failure of an idea depends not only on the goodness of the idea but also on the diffusion process. There are many examples in history of ideas that were dismissed at first and much later proven right. As many ideas spread through social contact, the social structure of individuals is likely to be determinant in the diffusion process. The present paper performs a theoretical study of the influence of social reinforcement on the diffusion of ideas in a population of human agents.

There is an ongoing debate in recent literature as to which social network structure is optimal in terms of diffusion [1]. The first strand of literature builds upon [2] "weak ties hypothesis". According to this idea, long ties between otherwise unconnected neighborhoods facilitate the spread of information, as they reduce the redundancy of the diffusion process [3].

The second strand of literature builds on the work by [4] and argues that close social structures promote trust, and thus facilitates information sharing and transmission. Thus, networks with overlapping neighborhoods (highly clustered networks) are better suited to promote diffusion [5].

The empirical evidence to support both theories is wide and strong. In recent works, Damon Centola has argued that none of them can be generalized to "whatever is diffused", but depends on whether the process is a simple or a complex contagion process [6], [7], [8], [9]. In simple contagions, only the first contact with an infected agent determines whether or not an agent is infected. In such a case, information in closed neighborhoods is redundant, and long ties can bridge distant neighborhoods and allow for information to travel through the network. In complex contagions, on the other hand, transmission depends on interactions with multiple infected agents. Thus, clustered neighborhoods are not redundant anymore, but provide with multiple sources of reinforcement that can promote transmission. Accordingly, they finds that complex contagion processes diffuse better in clustered networks like small-worlds [10] or lattices than in random networks [11].

In this paper we will argue that it is not only the nature of the diffusion process but the distribution of "incredulity" or resistance to contagion of agents, that determines the performance of different network structures. In order to do so, we build upon a percolation framework to study the interplay of individual preferences and social reinforcement, in order to have a theoretical benchmark that can help understanding the role of structural factors such as clustering in diffusion processes. We consider that ideas are diffused by word-of-mouth [12] by friends in a complex way. We find that for uniform distributions of incredulity, the strength of weak ties hypothesis can still apply for complex contagion processes. On the other hand, for incredulous populations the reinforcement mechanism is more important and clustered networks do better than random ones.

The structure of the paper is as follows. Section II introduces the basic model and the extension with social reinforcement. In Section III we introduce a different distribution of agents. Finally, in Section IV we present some conclusions.

II. Basic percolation model and social reinforcement extension

A. Basic percolation

In this article we study the diffusion process of new ideas on a population that presents a social network structure. Ideas are identified by their value, represented by a number $v \in [0,1]$. Agents are heterogeneous and they are characterized by a minimum quality requirement (MQR) for adopting a new idea. The higher the MQR -the more "incredulous" an agent is- the higher the value he requires of an idea in order to adopt it. The MQR of agents is a random variable which is uniformly distributed, $q \sim U[0,1]$. This modelling framework corresponds to the so-called percolation model [13].

In a percolation model of diffusion one agent adopts the new idea at any given time t (time is discrete) if the following three conditions are met:

- the agent has not adopted before t,
- the agent is informed, which only occurs if at least one neighbor has adopted at time $t-1$,
- the value of the idea is higher than the MQR of the agent, that is $q < v$.

Without a social structure the percolation model behaves as a well-mixed population of consumers. In a well-mixed population, agents are not embedded in a social network and they have perfect information. As soon as the idea enters the "market", the willing to adopt agents adopt it while the rest don't. As the MQR is uniformly distributed $q \sim U[0,1]$, a proportion $100 \times v_0\%$ of the population will adopt an idea of value $v_0 \in [0,1]$. This case can be represented in our model with a complete network, where every agent is connected to every other agent. In a complete network, a single early adopter will inform the whole population of agents about the existence of the idea.

B. Network structure

In a percolation setting, agents become informed of the existence of the idea through her neighbors. Thus, the structure of the social network where the agents are embedded can be determinant of the outcome of the process [6]. Previous studies have considered percolation processes in regular networks as a two dimensional lattice [14], [15], [16] or a completely random network [12] . These networks do not offer an accurate description of a social network [17], although their simplicity can be useful for their implementation and the interpretation of the results.

In this paper we propose the use of the small world algorithm [10] for the modelling of the social structure as in [18]. This provides with a family of networks, an interpolation between regular lattices and completely random networks. The algorithm starts with a regular ring lattice and rewires every link with probability μ. This parameter allows to fine tune the randomness of the network.

The small world algorithm produces a network structure that reproduces two well-known properties of social networks. On the one hand, they have a high clustering coefficient. That is to say, that the probability of two nodes to be connected together is higher if they share a mutual neighbor. This is a typical characteristic of social networks, where friendship groups are tight communities, and friends share many connections. On the other hand, small worlds have a low average path length. This is the so-called "six degrees of separation" theory introduced by [19], according to which every person in the world is separated from every other person by a very small number of connections such as friendship.

Varying the rewiring probability μ of the small world algorithm produces networks with varying average path length and clustering coefficient (Figure 1). The case with $\mu = 0$ is the one-dimensional regular lattice, and the case with $\mu = 1$ is the random network, also known as Poisson network or Erdos-Renyi model. The "typical" Small World is the one with rewiring probability $\mu = 0.01$, presenting an average path-length almost as low as the Poisson network, while still having a clustering coefficient which is comparable with the one-dimensional regular lattice.

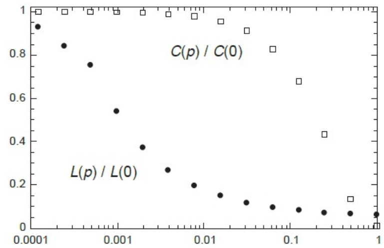


Fig. 1: Clustering coefficient $C(p)$ and average path-length $L(p)$ as a function of the rewiring probability in small world networks [10].

C. Social reinforcement

The difference between the basic percolation model and the social reinforcement extension lies in how the MQR of agents is calculated. Let q_t be the MQR of an agent at time t. In the basic percolation model this threshold remains constant over time, with $q_t = q_0 \ \forall t$. Thus, the number of adopting neighbors does not play any role in adoption decisions. Nothing changes for an agent if she knows about the new idea from one or many neighbors: the number of adopting neighbors does not have any weight, and additional adoptions are only redundant information. We extend this model by introducing a local social reinforcement effect. We include a new factor in the expression of the value of an idea, according to which decisions are influenced by the number of adopting neighbors. Adopting neighbors can "advocate" in favor of the idea, so as to increase the likelihood of its adoption.

The updated MQR is defined to satisfy the following hypothesis of the model. Let $q \in [0,1]$ be the MQR of an agent, $a \in \mathbb{N}$ the number of adopting neighbors and $\gamma \in [0,1]$ a parameter expressing the social reinforcement intensity. The functional form $f(q,a,\gamma)$ is chosen such that:

1) it is decreasing in the absolute number of adopting neighbors, $\frac{\partial f}{\partial a} < 0$;
2) it is decreasing in social reinforcement, $\frac{\partial f}{\partial \gamma} < 0$;
3) with only one neighbor adopting it is equal to the initial MQR q_0;
4) without social reinforcement ($\gamma = 0$) it is equal to the basic percolation model.

The first condition implies that neighbors give positive information about the idea: the more neighbors adopt, the easier it is for an agent to adopt. The second condition means that social reinforcement is a positive force for adoption. With the same number of adopting neighbors, the updated value of MQR will be lower for higher social reinforcement intensities, so adoption will be easier. The first decision to adopt for an agent is after the first adoption in her neighborhood. In order to compare our results with the benchmark percolation case, we need the MQR of agents to be their initial value with only one neighbor adopting (third condition). Finally, the fourth condition allows us to keep the benchmark percolation as a particular case of the extended model. The functional form in Equation (1) fulfills all four conditions.

$$\begin{aligned} q_t^i &= q_0^i \cdot \left(\frac{1}{\#\ neighbors\ of\ i\ that\ have\ adopted}\right)^{\gamma} \\ &= q_0^i \cdot \left(\frac{1}{a_t^i}\right)^{\gamma} \end{aligned} \quad (1)$$

D. Simulation results

In this section we study the percolation model extended with social reinforcement by mean of batch simulation experiments. For the social network structure, different instances of the small world model [10] are considered, which are identified by a rewiring probability $\mu \in \{0, 0.001, 0.01, 0.1, 1\}$. We consider $N = 10,000$ nodes representing potential adopters, with $k = 4$ neighbors on average. We simulate the model in different settings represented by the rewiring probability μ (network structure), the idea initial value v_0, and the social reinforcement intensity γ. The MQRs of agents are random draws from a uniform distribution, $q \sim U[0,1]$. For each setting we run $R = 50$ simulations, and look at the average value of the diffusion size together with its standard deviation across the different runs. In all simulations the diffusion process is initialized with 10 early adopters, the seeds of the simulation.

Results of the simulations are reported in Figure 2. Without social reinforcement ($\gamma = 0$), the social structure creates “information failures” compared to the well-mixed population with perfect information. Some willing to adopt agents never become informed of the existence of the idea because none of their neighbors have adopted it. Thus, the final diffusion size is lower than the linear demand (dashed line).

We first observe that in the diffusion regime of percolation (above the threshold represented by the sharp increase in diffusion size), the social reinforcement factor adds to the diffusion levels of the basic percolation model. Moreover, the number of adopters can even surpass the linear diffusion level of a well-mixed population. This is because with social reinforcement agents get to have a subjective valuation of the idea which is above its initial value v_0, and possibly above their minimum required quality even if the initial value was below it.

A second but possibly more important change is for the position of the percolation threshold. For the Poisson network ($\mu = 1$), an increasing social reinforcement intensity does very little, since the position of the threshold is almost unaffected across the different panels in Figure

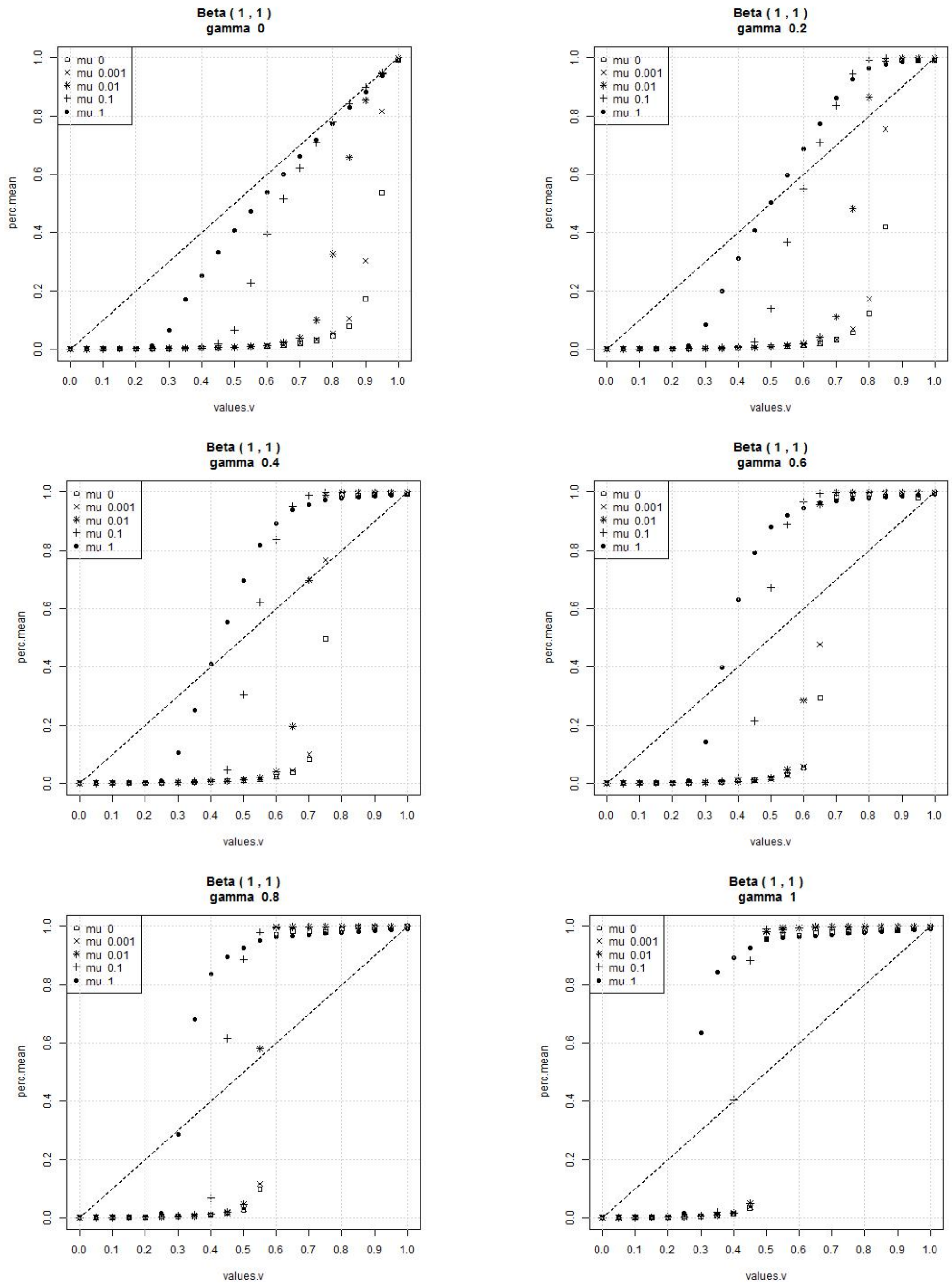


Fig. 2: Diffusion size in different small world networks for different initial values $v_0 \in [0, 1]$ of the diffusing idea (horizontal axis) in different conditions of social reinforcement intensity $\gamma \in [0, 1]$ (different panels). Reported values are averages over 50 simulation runs. The network size is $N = 10,000$ nodes, with 10 early adopters (seeds).

2. The opposite is true for the regular one-dimensional lattice and for Small World networks with $\mu = 0.001$ and $\mu = 0.01$, that see their thresholds moving substantially to lower values as γ increases. For instance, the typical Small World network with $\mu = 0.01$ has a threshold equal to 0.2 without social reinforcement,[1] which goes down to about 0.7 with $\gamma = 0.4$ and to 0.6 with $\gamma = 1$.

The thresholds to percolation do not just decrease, they also seem to change their nature. Without social reinforcement (Figure 2) the threshold from non-diffusion to diffusion regimes is a second order transition: there is a sharp but continuous change in the number of adopters. With social reinforcement (Figure 2), on the other hand, the threshold looks more like a first order transition: the number of adopters jumps from almost zero to almost full diffusion. This can be the result of a critical mass scenario. As soon as there is a sufficient number of adopters, the social reinforcement forces the process to cascade to complete diffusion. This effect only happens in highly clustered networks ($\mu = 0.01$ or lower). Without social reinforcement clustering hampers diffusion, since most links are redundant and cannot be used to reach new sources of information. With social reinforcement, though, another effect arises: shared friends may lead an agent to adoption by increasing her subjective value of an idea. Assume for instance that at a time t agent i, Margaret, sees Bill, one of her neighbors, adopting the idea. Still, the initial value of the idea is below Margaret's minimum required level, $v_0 < q_r^i$. At time $t+1$ another of her neighbors, Elinor, or agent j, adopts. This happens exactly because their common friend Bill had adopted the period before. Elinor had a lower minimum requirement level than Margaret, which happens to be such that $v_r^j < v_{t+1} < v_r^i$. Now, with two neighbors adopting, the value of the idea for Margaret becomes high enough as to be above her minimum requirement, $v_{t+2} > v_r^i$. This is how the triadic structure of their mutual friendship makes it possible for Margaret to adopt at a later stage, which would have not happened in a different social structure. Figure 3 shows an example of this dynamic.

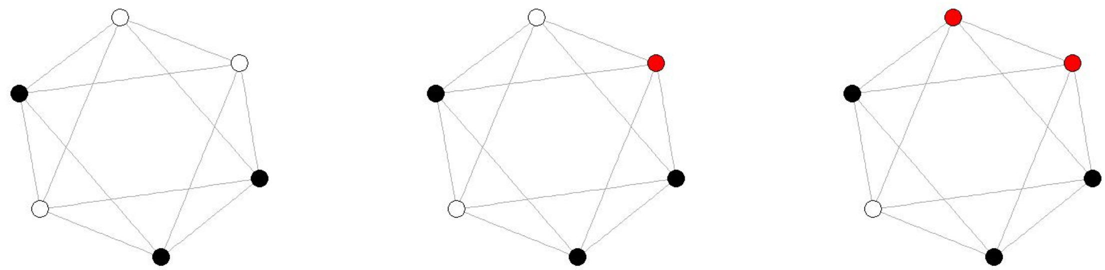
Finally, Figure 2 shows that increasing social reinforcement intensity reduces the differences between network structure. While without social reinforcement ($\gamma = 0$) there are differences in the final size of diffusion for $v_0 \in [0.3, 1]$ approximately, with a high social reinforcement ($\gamma = 1$) this range is reduced to $v_0 \in [0.3, 0.5]$. This result has important implications for policies aiming at introducing some new behavior or idea: when agents can be convinced by their friends, it is not so important to know the social network structure. In a well-mixed population, perfect information implies that every agent instantly knows about any new idea. In a network setting, this situation is represented by a fully connected network, where every agent is neighbor of every other agent. In this case, social reinforcement would lead to full diffusion even for small values of the idea.[2] Thus, it is important to know that there is some kind of network structure in the process. It is not so important, however, which structure this is as long as it is not a perfect information setting.

III. Non-uniform distributions

Most studies on complex propagation consider that agents are homogeneous in their resistance to contagion [6], [11]. In the previous section, we relax this assumption by assuming a uniform distribution of MQRs. Nonethe-

[1] The theoretical value is about 0.82, according to [20].

[2] If an idea of value v_o is introduced, a proportion v_o of the N agents would immediately adopt it, that is a total of $N \cdot v_o$ agents. In the following step, the MQR of the remaining agents has been decreased by $\frac{1}{N v_o}^{\gamma}$: at the end of the second step, $v_o(v_o N)^{\gamma}$ agents have adopted. The process continues so that after the s step, $v_o^{1+s\gamma} N^{s\gamma}$ agents have adopted. If $N > frac1v_o$, then $lim_s(v_o^{1+s\gamma} N^{s\gamma} = \inf$, so the process reaches full diffusion.



(a) The white nodes represent the willing to adopt neighbors.

(b) A willing to adopt agent adopts (red node).

(c) The clustering links make social reinforcement more intense for non-willing to adopt agents.

Fig. 3: The effect of social reinforcement on clustered neighborhoods

less, the outcome of the diffusion process is highly dependent on the specific distribution considered.

The marginal effect on the MQR of an having an additional neighbor adopting is described in Equation (**??**). The first adopting friends induce a large decrease in the MQR, while after a large number of friends have adopted the influence of an additional adopting neighbor is negligible. Moreover, the effect on one more friend adopting is larger for large values of q_o^i. Thus, in this section we will concentrate on a case where agent have high initial MQRs.

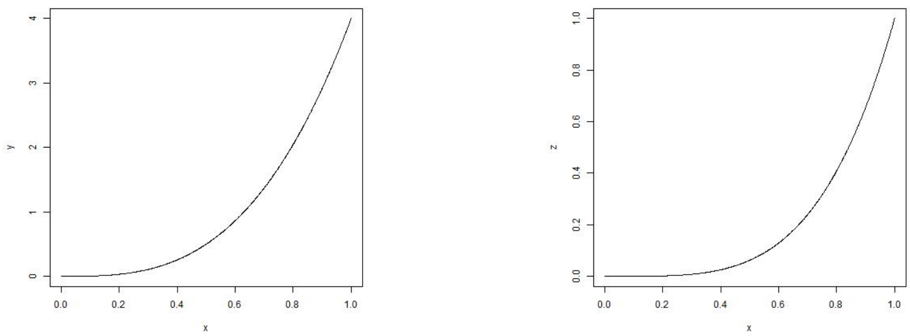
$$
\begin{aligned}
q_t^i &= q_0^i(1/a_t^i)^\gamma \rightarrow \Delta q_t^i \\
&= q_0^i \Delta (a_t^i)^{-\gamma} \\
&= q_0^i(-\gamma (a_t^i)^{-gamma-1} \Delta (a_t^i) \\
&= -\frac{q_0^i \gamma}{(a_t^i)^{\gamma+1}} \Delta a_t^i
\end{aligned} \tag{2}
$$

As the effect of social reinforcement is higher upon more reluctant or incredulous agents, we analyze here the diffusion process in an incredulous population. Figure 4 shows the density of a $Beta(\alpha = 4, \beta = 1)$ distribution. All values drawn from this distribution will be bounded to $[0, 1]$ as in the uniform distribution $U[0, 1]$, although they will be biased towards high values close to one. That way, a distribution $Beta(4, 1)$ represents a population where most people are incredulous, or unwilling to adopt the idea, and a few people are enthusiastic early adopters. This is a realistic population that would provide an s-shaped adoption curve over time [21].

A. Simulation results

As in the previous section, we use batch simulations to compare the behavior of the diffusion process under different conditions. We compare five network structures from the samll world algorithm [10] with rewiring probabilities $\mu \in \{0, 0.001, 0.01, 0.1, 1\}$, $N = 10,000$ nodes and $k = 4$ initial neighbors. The MQRs of agents are now drawn from a $q \sim Beta(4, 1)$ distribution. For every setting of value of the idea $v_o \in [0, 1]$, social pressure $\gamma \in [0, 1]$ and rewiring probability μ we study the mean diffusion size over $R = 50$ runs with 10 seeds or initial adopters.

Results of the simulations are depicted in Figure 5. Increasing the social reinforcement intensity γ increases the number of adopters, as some of the unwilling to adopt are convinced. It also decreases the percolation thresholds, the minimum value of the idea v_o that gets



(a) Density function

(b) Cummulative distribution

Fig. 4: Probability distributions of a $Beta(4,1)$.

some diffusion. This effect is more accused for the more regular networks, due to the reinforcement of clustering links. In the random networks, however, the thresholds are more stable. Thus, as the social reinforcement intensity γ increases, the graphics for the different networks overlap as in Figure 5, for $\gamma = 0.4$. If we continue to increase γ, the clustered networks will overperform the random Poisson network.

IV. Conclusions

Introducing social reinforcement in a percolation model of diffusion adds to the size of diffusion. In the case of ideas that would otherwise not be diffused, social reinforcement allows for some spreading in the population. It also reduces the differences between network structures. Without social reinforcement, clustering links are redundant: if the number of ties is limited, they restrict the access to new sources of information. Nonetheless, when the opinion of neighbors can influence the adoption process, clustering links can force agents to cascade to adoption.

In simple propagations no social reinforcement is present ($\gamma = 0$) and thus the size of diffusion is determined by the number of willing to adopt agents that the idea can reach. That is to say, the diffusion is determined by the dimensionality of the network, how many agents can be reached with every new step. As random networks have the higher dimensionality, they are the most efficient structures to spread an idea. In the small world algorithm, clusters come at the expenses of bridges: the more clustered the network is, the lower its dimensionality as clustering links are redundant.

For complex propagations clustering links are not redundant anymore. Indeed, they provide an additional support for the social reinforcement mechanism. Once a first neighbor has adopted, the probability that a second neighbor adopts increases with clustering coefficient. In the limit case, a random network, the probabilities of different neighbors adopting are independent. Thus, introducing social reinforcement affects the diffusion in the random network vaguely. It increases the number of adopters, as some of the unwilling to adopt are convinced, but leaves the percolation thresholds essentially unmoved. On the other hand, the interaction of social reinforcement with the structure of highly clustered networks alters both the number of adopters and the thresholds of the shift from a non-diffusion to a diffusion regime. Moreover, there appears to be a change in the nature of these thresholds, from a second order transition to a first order (discontinuous) transition. The interplay of clustered networks

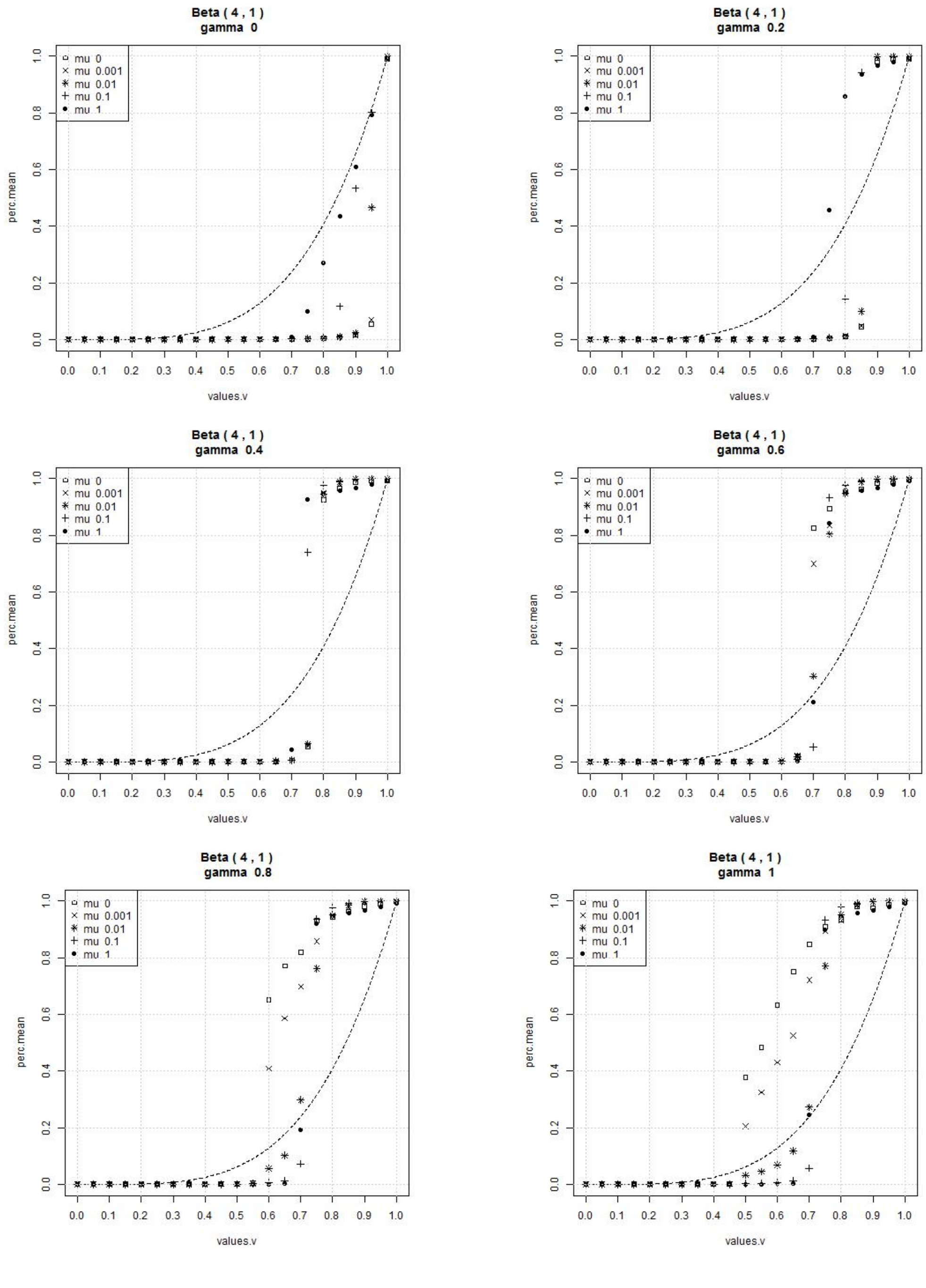


Fig. 5: Diffusion size in different small world networks for different initial values $v_0 \in [0, 1]$ of the diffusing idea (horizontal axis) in different conditions of social reinforcement intensity $\gamma \in [0, 1]$ (different panels). Reported values are averages over 50 simulation runs. The network size is $N = 10,000$ nodes, with 10 early adopters (seeds). The initial MQR values in the population follows a $Beta(4, 1)$ distribution.

decreasing their thresholds while random ones remain more stable results in an homogenization of the results for the different networks. For high intensities of social reinforcement, it is important to know that there is a social network underlying the process of diffusion, but not so important to know which network it is. Nonetheless, even with this uniformization of the network structures random networks still come as the most efficient structures to enhance diffusion.

In this setting, random networks get higher shares of diffusion both for simple and complex propagations, contrary to the findings of [6], [9], [8]. Nonetheless, changing the distribution of "incredulity" throughout the population of agents can confirm their results. Our study confirms that clustering can be favorable or harmful for diffusion, depending on the setting. Nonetheless, the determinant of which network structure is more efficient for spread is not only the nature of the process (a complex or a simplex propagation), but also the characteristics of the population in which it diffuses.

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